SOLUCIONES MODELO MATEMATICAS I (PD) - 5ª EV - PRUEBA 2 (con CALCULADORA)

1) The first three terms of an arithmetic sequence are 5 , 6.7 , 8.4 . **[2+2+2]**

a) Find the common difference.

b) Find the 28th term of the sequence.

c) Find the sum of the first 28 terms.

a)

e.g. subtracting terms, using sequence formula

$d = 1.7$ *A1* *N2*

b)

e.g. $5 + 27(1.7)$

28th term is 50.9 (exact)

c)

e.g. $S_{28} = \frac{28}{2}(2(5) + 27(1.7))$, $\frac{28}{2}(5 + 50.9)$

$S_{28} = 782.6$ (exact) [782, 783]

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2) Consider the expansion of $\left(2x^3 + \frac{b}{x}\right)^8 = 256x^{24} + 3072x^{20} + \ldots + kx^0 + \ldots$. **[3+3]**

a. Find b.

b. Find k.

a)

valid attempt to find term in x^{20}

e.g. $\binom{8}{1}(2^7)(b)$, $(2x^3)^7\left(\frac{b}{x}\right) = 3072$

correct equation

e.g. $\binom{8}{1}(2^7)(b) = 3072$

$b = 3$

b)

e.g. 7th term, $r = 6$

correct expression

e.g. $\binom{8}{6}(2x^3)^2\left(\frac{3}{x}\right)^6$

$k = 81648$ (accept 81600)

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3) Draw a draft of two functions that:

a)

$\lim_{x \to -\infty} f(x) = -1$ $\quad \lim_{x \to 3^-} f(x) = +\infty$

$\lim_{x \to -2^-} f(x) = -\infty$ $\quad \lim_{x \to 3^+} f(x) = -\infty$

$\lim_{x \to -2^+} f(x) = +\infty$ $\quad \lim_{x \to +\infty} f(x) = -1$

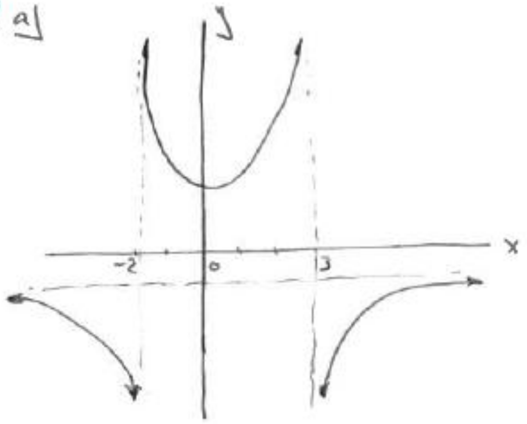
b)

$\lim_{x \to 2^-} f(x) = -3$ $\quad \lim_{x \to +\infty} f(x) = -\infty$

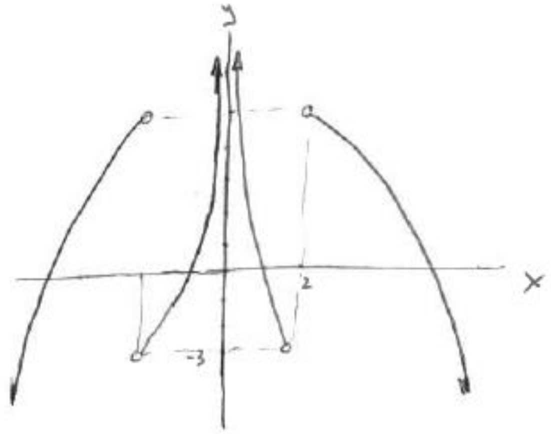
$\lim_{x \to 2^+} f(x) = 5$ $\quad f(x)$ OY-symmetric

$\lim_{x \to 0^+} f(x) = +\infty$

3) a)

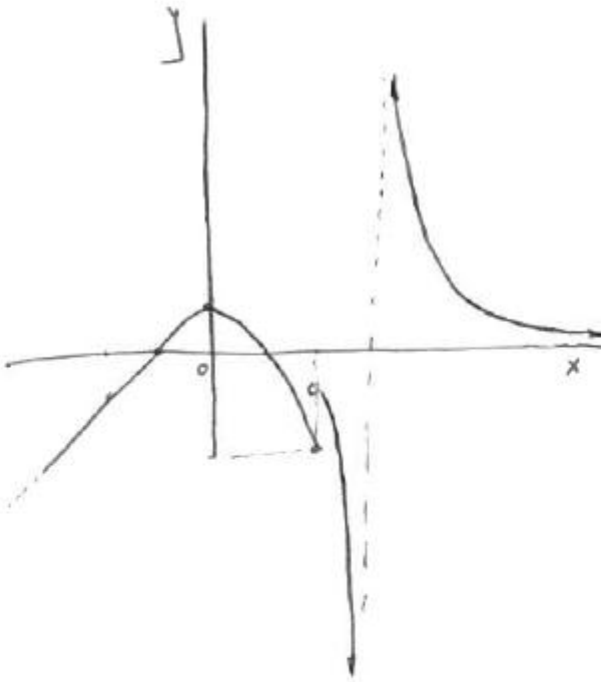


b)

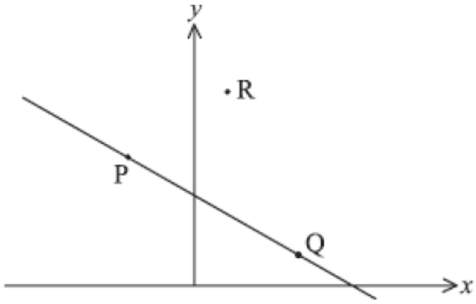


4. Obtain the Domain and draft the grapfic: $f(x) = \begin{cases} x+1 & \text{Si } x \le -1 \\ 1-x^2 & \text{Si } -1 < x \le 2 \\ \dfrac{1}{x-3} & \text{Si } x > 2 \end{cases}$

4. $x-3=0 \to x=3$ $\quad \text{dom} f = (-\infty,-1] \cup (-1,2] \cup (2,+\infty) - \{3\} = \boxed{\mathbb{R} - \{3\}}$



5) En el diagrama siguiente se muestran los puntos P(−2, 4), Q(3, 1) y R(1, 6).



(a) Halle el vector $\overrightarrow{PQ}$.

(b) Halle una ecuación vectorial de la recta que pasa por R y es paralela a la recta (PQ).

5) a) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

b) $\boxed{\vec{r} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} + t \cdot \begin{pmatrix} 5 \\ -3 \end{pmatrix}}$

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6) La siguiente figura muestra un círculo de centro O y radio 12 cm. La cuerda AB determina un ángulo central de 75°. Las tangentes a la circunferencia en A y en B se encuentran en P.

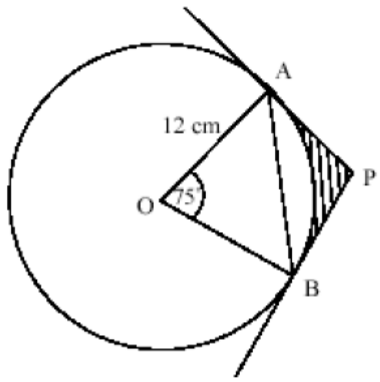
a) Halla el área del sector OAB

b) Halla el área del triángulo OAB

c) Demuestra que $AB = 12\sqrt{2(1-\cos 75°)}$

d) Halla el área del triángulo ABP

e) Halla el área de la región sombreada



a) $75° = \dfrac{75 \cdot \pi}{180} = \dfrac{5\pi}{12}\, rad \rightarrow Area\ Sector = \dfrac{12^2 \cdot 5\pi/12}{2} = \boxed{30\pi\ cm^2}$

b) $Area\ Triángulo\ OAB = \dfrac{12 \cdot 12 \cdot sen75°}{2} \approx \boxed{69{,}5\, cm^2}$

c) $AB = \sqrt{12^2 + 12^2 - 2 \cdot 12 \cdot 12 \cdot \cos 75°} = \sqrt{12^2(1+1-2\cdot\cos 75°)} = 12\sqrt{2(1-\cos 75°)} \approx \boxed{14{,}6cm}$

d) $180° - 75° = 105° \rightarrow O\hat{A}B = O\hat{B}A = \dfrac{105}{2} = 52{,}5° \rightarrow B\hat{A}P = A\hat{B}P = 90° - 52{,}5° = 37{,}5° \rightarrow A\hat{P}B = 180° - 2 \cdot 37{,}5° = 105°$

$AB^2 = AP^2 + PB^2 - 2 \cdot AP \cdot PB \cdot \cos 105°$

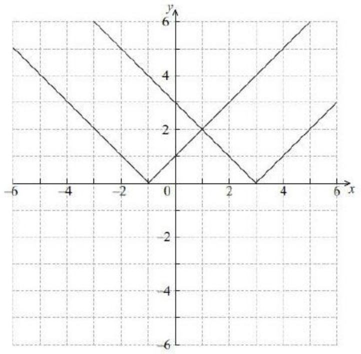
Como $AP = PB \rightarrow AB^2 = AP^2 + AP^2 - 2 \cdot AP \cdot AP \cdot \cos 105° \rightarrow AB^2 = AP^2(2 - 2\cdot\cos 105°) \rightarrow$

$\rightarrow AP^2 = \dfrac{AB^2}{2 - 2\cdot\cos 105°} \Rightarrow AP = \dfrac{AB}{\sqrt{2-2\cdot\cos 105°}} = \dfrac{14{,}6}{\sqrt{2-2\cdot\cos 105°}} \approx 9{,}20\, cm$

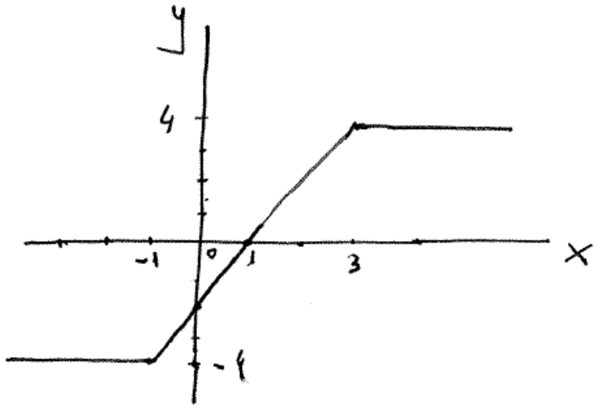
$Area\ Triángulo\ ABP = \dfrac{AB \cdot AP \cdot senB\hat{A}P}{2} = \dfrac{14{,}6 \cdot 9{,}20 \cdot sen37{,}5°}{2} \approx \boxed{40{,}9\, cm^2}$

e) $Area\ Sombreada = Area\ Triángulo\ OAB + Area\ Triángulo\ ABP - Area\ Sector \approx \boxed{16{,}2\, cm^2}$

7) The graphs of $y=|x+1|$ and $y=|x-3|$ are shown below.



Let $f(x)=|x+1|-|x-3|$. Draw the graph of $y=f(x)$ on the same grid



8) Hallar el área del recinto limitado por la gráfica de la función

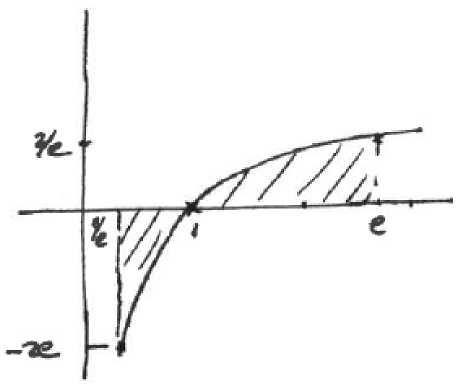
$$f(x) = 2\,\frac{\ln x}{x}$$

el eje de abscisas y las rectas $x = 1/e$ y $x = e$.
Razona la respuesta.

8) $\left.\begin{array}{l} y = 2\frac{\ln x}{x} \\ y = 0 \end{array}\right\}$ $0 = 2\frac{\ln x}{x} \Rightarrow \ln x = 0 \Rightarrow x = 1$

$y = \frac{2\ln x}{x}$ Dominio $= (0, +\infty)$

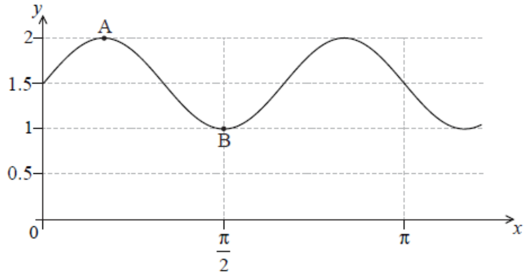
x	y
1	0
e	2/e
1/e	-2e



$$\text{Area} = -\int_{1/e}^{1} 2\frac{\ln x}{x}\,dx + \int_{1}^{e} 2\frac{\ln x}{x}\,dx = -\ln^2 x\Big|_{1/e}^{1} + \ln^2 x\Big|_{1}^{e} = -\ln^2 1 + \ln^2 \tfrac{1}{e} + \ln^2 e - \ln^2 1 =$$

$$= -0 + (-1)^2 + 1^2 - 0 = \boxed{2}$$

9) The following diagram shows part of the graph of $y = p\sin(qx) + r$.



The point $A\left(\frac{\pi}{6}, 2\right)$ is a maximum point and the point $B\left(\frac{\pi}{2}, 1\right)$ is a minimum point. Find the value of

(a) p; *[2]*

(b) r; *[2]*

(c) q. *[3]*

(a) valid approach

eg $\frac{2-1}{2}$, $2-1.5$

$p = 0.5$

(b) valid approach

eg $\frac{1+2}{2}$

$r = 1.5$

(c) **METHOD 1**

valid approach (seen anywhere)

eg $q = \frac{2\pi}{\text{period}}$, $\frac{2\pi}{\left(\frac{2\pi}{3}\right)}$

period $= \frac{2\pi}{3}$ (seen anywhere)

$q = 3$

METHOD 2

attempt to substitute one point and **their** values for p and r into y

eg $2 = 0.5\sin\left(q\frac{\pi}{6}\right) + 1.5$, $\frac{\pi}{2} = 0.5\sin(q1) + 1.5$

correct equation in q

eg $q\frac{\pi}{6} = \frac{\pi}{2}$, $q\frac{\pi}{2} = \frac{3\pi}{2}$

$q = 3$

METHOD 3

valid reasoning comparing the graph with that of $\sin x$

eg position of max/min, graph goes faster

correct working

eg max at $\frac{\pi}{6}$ not at $\frac{\pi}{2}$, graph goes 3 times as fast

$q = 3$

10. Consider the triangle PQR where $\hat{QPR} = 30^\circ$, $PQ = (x+2)$ cm and $PR = (5-x)^2$ cm, where $-2 < x < 5$.

(a) Show that the area, A cm^2, of the triangle is given by $A = \frac{1}{4}\left(x^3 - 8x^2 + 5x + 50\right)$.

(b) (i) State $\frac{dA}{dx}$.

(ii) Verify that $\frac{dA}{dx} = 0$ when $x = \frac{1}{3}$.

(c) (i) Find $\frac{d^2A}{dx^2}$ and hence justify that $x = \frac{1}{3}$ gives the maximum area of triangle PQR.

(ii) State the maximum area of triangle PQR.

(iii) Find QR when the area of triangle PQR is a maximum.

10. (a) use of $A = \frac{1}{2}qr\sin\theta$ to obtain $A = \frac{1}{2}(x+2)(5-x)^2 \sin 30^\circ$

$= \frac{1}{4}(x+2)(25 - 10x + x^2)$

$A = \frac{1}{4}(x^3 - 8x^2 + 5x + 50)$

(b) (i) $\frac{dA}{dx} = \frac{1}{4}\left(3x^2 - 16x + 5\right) = \frac{1}{4}(3x-1)(x-5)$

(ii) **METHOD 1**

EITHER

$\frac{dA}{dx} = \frac{1}{4}\left(3\left(\frac{1}{3}\right)^2 - 16\left(\frac{1}{3}\right) + 5\right) = 0$

OR

$\frac{dA}{dx} = \frac{1}{4}\left(3\left(\frac{1}{3}\right) - 1\right)\left(\left(\frac{1}{3}\right) - 5\right) = 0$

THEN

so $\frac{dA}{dx} = 0$ when $x = \frac{1}{3}$

METHOD 2

solving $\frac{dA}{dx} = 0$ for x

$-2 < x < 5 \Rightarrow x = \frac{1}{3}$

so $\frac{dA}{dx} = 0$ when $x = \frac{1}{3}$

METHOD 3

a correct graph of $\frac{dA}{dx}$ versus x

the graph clearly showing that $\frac{dA}{dx} = 0$ when $x = \frac{1}{3}$

so $\frac{dA}{dx} = 0$ when $x = \frac{1}{3}$

(c) (i) $\dfrac{d^2A}{dx^2}=\dfrac{1}{2}(3x-8)$

for $x=\dfrac{1}{3}$, $\dfrac{d^2A}{dx^2}=-3.5\,(<0)$

so $x=\dfrac{1}{3}$ gives the maximum area of triangle PQR

(ii) $A_{max}=\dfrac{343}{27}(=12.7)(cm^2)$

(iii) $PQ=\dfrac{7}{3}$ (cm) and $PR=\left(\dfrac{14}{3}\right)^2$ (cm)

$QR^2=\left(\dfrac{7}{3}\right)^2+\left(\dfrac{14}{3}\right)^4-2\left(\dfrac{7}{3}\right)\left(\dfrac{14}{3}\right)^2\cos 30^\circ$

$=391.702\ldots$

$QR=19.8$ (cm)

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11) Hallar el área del recinto limitado por los ejes de coordenadas, la recta y = 2 y la curva de ecuación

$$y=\sqrt{x-2}\,.$$

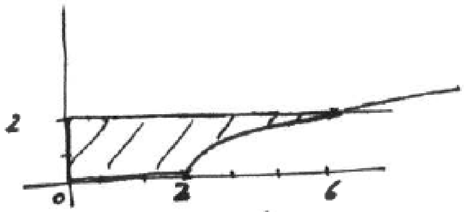
Razona la respuesta.

11) $\left.\begin{array}{l} y=\sqrt{x-2} \\ x=0 \end{array}\right\}$ Sin intersección

$\left.\begin{array}{l} y=\sqrt{x-2} \\ y=2 \end{array}\right\} \Rightarrow 2=\sqrt{x-2}\ ;\ 4=x-2\ ;\ x=6$

$y=\sqrt{x-2}$ Dominio $=[2,+\infty)$

x	y
2	0
6	2



$$\text{Area}=2\cdot 2+\int_2^6\left(2-\sqrt{x-2}\right)dx=4+\int_2^6\left(2-(x-2)^{1/2}\right)dx=4+\left(2x-\frac{(x-2)^{1/2}}{3/2}\right)\Big|_2^6=$$

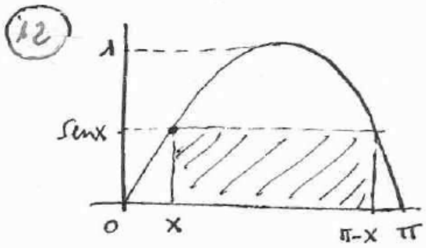
$$=4+\left(12-\tfrac{2}{3}4^{3/2}\right)-(4-0)=12-\frac{16}{3}=\boxed{\frac{20}{3}}$$

12 Se dibuja un rectángulo cuyos vértices inferiores se encuentran en el eje OX y cuyos vértices superiores se encuentran en la curva $y = senx$, siendo $0 \le x \le \pi$

a) Escriba una expresión para el área del rectángulo

b) Halle el área máxima del rectángulo

12



$$Área = (\pi - 2x)\,sen\,x \qquad x \in (0, \pi/2)$$

$$\frac{dA}{dx} = -2\,sen\,x + (\pi - 2x)\cos x$$

$$\frac{dA}{dx} = 0 \rightarrow -2\,sen\,x + (\pi - 2x)\cos x = 0 \rightarrow$$

$$\rightarrow 2\,sen\,x = (\pi - 2x)\cos x \rightarrow 2\,tg\,x = \pi - 2x \Rightarrow \boxed{x = 0'710246}$$

Resuelta con calculadora gráfica

$$Área\ Máxima = (\pi - 2 \cdot 0'71)\,sen\,0'71 = \boxed{0'021}$$